

# Product posets and causal automorphisms of the plane

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## Abstract

A simple characterization of the causal automorphisms of 1+1 Minkowski spacetime is given.

## 1 Introduction

In this work by poset  $P$  we mean a partially ordered set, that is a set endowed with a reflexive, transitive and antisymmetric relation. An order automorphism of  $P$  is a surjective map  $f : P \rightarrow P$  with the property that  $p \leq q \Leftrightarrow f(p) \leq f(q)$ . An order automorphism is necessarily injective. Indeed,  $f(p) = f(q)$  implies “ $f(p) \leq f(q)$  and  $f(q) \leq f(p)$ ” from which we deduce “ $p \leq q$  and  $q \leq p$ ” and hence  $p = q$ . Given an order automorphism  $f$  the inverse  $f^{-1}$  is also an order automorphism.

A spacetime  $(M, g)$  endowed with the causal order  $\leq$ , i.e.  $p \leq q$  if there is a future directed causal curve connecting  $p$  to  $q$  or  $p = q$ , is called causal if  $(M, \leq)$  is a poset. The order automorphisms of a causal spacetime endowed with the causal relation are then called causal automorphisms. We shall not demand that these maps be continuous but continuity will follow.

The causal automorphisms of Minkowski  $n + 1$  spacetime for  $n \geq 2$  have been shown by Alexandrov [1, 3, 2] and Zeeman [13] to be generated by the inhomogeneous Lorentz group and dilatations. Under the assumption of differentiability this result was previously obtained by Liouville and Lie [4] so that Alexandrov-Zeeman theorem follows from Liouville-Lie theorem by noting that every bijective map between strongly causal spacetimes which sends null geodesics into null geodesics and conversely is smooth [6, 7].

Recently, Do-Hyung Kim has investigated the causal automorphism of Minkowski 1+1 spacetime, namely  $M = \mathbb{R}^2$  with coordinates  $(x^+, x^-)$  and metric  $g = -dx^+ dx^-$  (we use preferably light cone coordinates  $x^\pm = t \pm x$ ). With respect to the higher dimensional case here we have the complication that the causal automorphism need not be smooth. Kim reaches a simple and intuitive result. Nevertheless, his proof is scattered over three papers [8, 9, 10], requires several propositions, and uses a tool of

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“causally admissible system” developed by him and connected to the representability of spacetime points with compact sets on a Cauchy hypersurface.<sup>1</sup> Given the simplicity of the final result it is natural to wonder whether it could be obtained through a simple direct proof. We provide such a proof. In the end we obtain

**Theorem 1.1.** *The causal automorphism of Minkowski 1+1 spacetime are generated by the space reflection  $(x^+, x^-) \rightarrow (x^-, x^+)$  and by the maps  $(x^+, x^-) \rightarrow (f(x^+), g(x^-))$  where  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  are increasing homeomorphisms of the real line.*

Note that we do not demand that the causal automorphism be continuous but the continuity follows. The fact that the space reflection  $(x^+, x^-) \rightarrow (x^-, x^+)$  is a causal automorphism can be easily checked. Also the fact that the map  $(x^+, x^-) \rightarrow (f(x^+), g(x^-))$  is a causal automorphism whenever  $f$  and  $g$  are increasing homeomorphisms of the real line can be easily verified. The proof of the theorem is based on the observation that Minkowski spacetime, as it is evident in light cone coordinates, is the product poset of the real line (with the usual order) with itself. Recall that if  $A$  and  $B$  are two posets then  $A \times B$  can be given the product order:  $(a, b) \leq (a', b')$  if  $a \leq a'$  and  $b \leq b'$ . The causal order has indeed this structure because  $(x^+, x^-) \leq (y^+, y^-)$  iff  $x^+ \leq y^+$  and  $x^- \leq y^-$ .

*Proof.* Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a causal automorphism (in light cone coordinates). Let  $O = (0, 0)$  and define  $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by  $G(p) = F(p) - F(O)$  then since translations preserve the causal relation we have that  $G$  is a causal automorphism that maps  $O$  to itself. In a spacetime the horismos relation is the set  $E^+ = J^+ \setminus I^+$ , in particular  $(x, y) \in E^+$  if and only if there is a future directed achronal lightlike geodesic connecting  $x$  to  $y$  (see [5]). In Minkowski spacetime we have  $(p, q) \in E^+$  if and only if  $J^+(p) \cap J^-(q)$  is totally ordered by the causal relation  $\leq$  (for generic causal spacetimes a similar characterization of  $E^+$  holds, see [11] [12, Theor. 3.9]). Since  $G$  and  $G^{-1}$  preserve the causal relation  $\leq (= J^+)$  and  $E^+$  can be completely characterized through it, it follows that  $(p, q) \in E^+$  if and only if  $(G(p), G(q)) \in E^+$ .

The axes passing through  $O$  form the set  $E := E^+(O) \cup E^-(O)$  and since  $O$  is mapped into itself we find that  $E$  is mapped into itself. Let us consider any two distinct points  $p, q \neq O$  in one of the axis passing through  $O$ . We can assume without loss of generality that  $J^+(p) \cap J^-(q) \neq \emptyset$  otherwise we exchange  $p$  and  $q$ . Since  $J^+(p) \cap J^-(q)$  is totally ordered by  $\leq$  the images  $G(p), G(q)$ , must satisfy  $J^+(G(p)) \cap J^-(G(q))$  is non-empty and totally ordered by  $\leq$  which is impossible if  $G(p)$  and  $G(q)$  do not lie on the same axis. We conclude that each axis passing through  $O$  is sent into another axis passing through  $O$  with a possible exchange of axes. It is trivial that the space reflection  $(x^+, x^-) \rightarrow (x^-, x^+)$  is a causal automorphism and thus by applying it once if necessary, we can always assume that the axes do not get interchanged by  $G$ .

Now, consider the axis whose elements are given by  $(x^+, 0)$ . This axis is mapped by  $G$  into itself, thus there is a function  $u : \mathbb{R} \rightarrow \mathbb{R}$  such that  $(x^+, 0) \rightarrow (u(x^+), 0)$ .

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<sup>1</sup>Kim essentially uses the light cone coordinates  $x \pm t$  instead of  $t \pm x$ , as a consequence the statement of his result needs to contemplate a case in which certain homeomorphisms are decreasing, a complication that does not appear in our formulation. Also note that he works on  $\mathbb{R}^2$  with coordinates  $(x, t)$  not  $(t, x)$ .

Analogously, there is a function  $v : \mathbb{R} \rightarrow \mathbb{R}$  such that  $(0, x^-) \rightarrow (0, v(x^-))$ . The function  $u$  is increasing because if  $a < b$  then, as  $(a, 0) \leq (b, 0)$ , we have  $(u(a), 0) \leq (u(b), 0)$  and hence  $u(a) \leq u(b)$ . The possibility  $u(a) = u(b)$  is ruled out for otherwise  $(u(a), 0) = (u(b), 0)$  in contradiction with the injectivity of  $G$ . Analogously,  $v$  is increasing.

Let us prove that  $G((x^+, x^-)) = (u(x^+), v(x^-))$ . Indeed,  $(x^+, 0) \leq (x^+, x^-)$  which implies after application of  $G$ ,  $(u(x^+), 0) \leq G((x^+, x^-))$ . Analogously, from  $(0, x^-) \leq (x^+, x^-)$  we get  $(0, v(x^-)) \leq G((x^+, x^-))$ . Therefore, we deduce that  $G((x^+, x^-)) \in J^+((u(x^+), 0)) \cap J^+((0, v(x^-))) = J^+((u(x^+), v(x^-)))$  which means that  $(u(x^+), v(x^-)) \leq G((x^+, x^-))$ . Now consider the point  $r' = (u(x^+), v(x^-))$ , since  $F$  is surjective,  $G$  is surjective thus there is  $r \in \mathbb{R}^2$  such that  $G(r) = r'$ . Moreover,  $r' \leq G((x^+, x^-))$  thus, as  $G^{-1}$  is a causal automorphism,  $r \leq (x^+, x^-)$ . However,  $(u(x^+), 0) \leq (u(x^+), v(x^-)) = r'$  and  $(0, v(x^-)) \leq (u(x^+), v(x^-)) = r'$  thus applying  $G^{-1}$  we get  $(x^+, 0) \leq r$  and  $(0, x^-) \leq r$  from which it follows  $(x^+, x^-) \leq r$ . We conclude that  $r = (x^+, x^-)$  and hence that  $G((x^+, x^-)) = (u(x^+), v(x^-))$ .

From this expression since  $G$  is surjective it follows that both  $u$  and  $v$  are surjective, and since  $u$  and  $v$  are surjective and increasing they are continuous. Moreover, being bijective and continuous they are homeomorphisms of the real line (invariance of domain theorem). As a final step define  $f = u + F(O)^+$  and  $g = v + F(O)^-$ .  $\square$

Let  $B$  denote the group of all causal automorphisms of the Minkowski plane and let  $N$  be the subgroup of causal automorphisms made of maps  $(x^+, x^-) \rightarrow (f(x^+), g(x^-))$  where  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  are increasing homeomorphisms of the real line. Denote with  $S_2$  the symmetric group of permutations of  $\{x^+, x^-\}$  which is also a subgroup of all causal automorphisms. It can be easily checked that  $N$  is a normal subgroup of  $B$ . Indeed, since we have proved that  $B = NS_2$  we have only to show that for  $n \in N$  and  $p \in S_2$ , with  $p$  the non trivial transposition, we have  $pn p \in N$ . Indeed, let  $n$  be the map  $(x^+, x^-) \rightarrow (f(x^+), g(x^-))$ , then  $np$  is the map  $(x^+, x^-) \rightarrow (f(x^-), g(x^+))$  and  $pn p$  is the map  $(x^+, x^-) \rightarrow (g(x^+), f(x^-))$  which belongs to  $N$ . As a consequence

**Corollary 1.2.** *The group of causal automorphism of the Minkowski plane is*

$$Hom_{\leq}(\mathbb{R})^2 \rtimes S_2$$

*namely the semidirect product between  $Hom_{\leq}(\mathbb{R})^2$  and  $S_2$  where  $Hom_{\leq}(\mathbb{R})$  is the group of order preserving homeomorphism of the real line.*

## 2 Conclusions

We have given a simple characterization of causal automorphisms of Minkowski 1+1 spacetime. Theorem 1.1 can be generalized, with very minor changes, to the case of a poset  $P$  which is the cartesian  $k$ -product,  $k \in \mathbb{N}$ , of a totally ordered poset  $X$ . Nevertheless, the spacetime interpretation of the product holds only in the case  $k = 2$  considered here.

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